

Symmetric Spaces in Supergravity

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Abstract

We exploit the relation among *irreducible Riemannian globally symmetric spaces (IRGS)* and *supergravity theories* in 3, 4 and 5 space-time dimensions. *IRGS* appear as scalar manifolds of the theories, as well as *moduli spaces* of the various classes of solutions to the classical extremal black hole *Attractor Equations*. Relations with *Jordan algebras of degree three and four* are also outlined.

1 Introduction

The aim of this contribution, devoted to the 70th birthday of Prof. Raja Varadarajan, is to give some examples of interplay among some mathematical objects, *Riemannian symmetric spaces*, and physical theories such as the supersymmetric theories of gravitation, usually called *supergravities*.

Symmetric spaces occur as *target spaces* of the *non-linear sigma models* which encode the dynamics of scalar fields, related by supersymmetry to some spin- $\frac{1}{2}$ and spin- $\frac{3}{2}$ fermion fields, the latter called *gravitinos*, the gauge fields of *local* supersymmetry.

Many *supergravities* provide a unique (classical) extension of the Einstein-Hilbert action of General Relativity. By denoting with n the number of supersymmetries (or equivalently the number of - real - components of suitably defined spinor *supercharges*), this holds for $n > 16$. In such a case, the *non-linear sigma model* of scalars is *unique*, and the dimension of the *symmetric space* $\frac{G}{H_{\mathcal{R}}}$ counts the number of scalar fields of the gravity multiplet. The isometry group $H_{\mathcal{R}}$ is nothing but the \mathcal{R} -symmetry of the \mathcal{N} -extended supersymmetry algebra, where \mathcal{N} is the number of *supercharges*. The *non-compact* global isometry group G is uniquely determined by the number of scalar fields and by the fact that G is a *non-compact, real form* of a *simple* (finite-dimensional) Lie group $G_{\mathbb{C}}$, whose *maximal compact subgroup* (*mcs*, with *symmetric embedding*, understood throughout) is $H_{\mathcal{R}}$. In $d = 3, 4$ and 5 space-time dimensions (which are the only ones we deal with in the present contribution) the \mathcal{R} -symmetry is $SO(\mathcal{N})$, $U(\mathcal{N})$ and $USp(\mathcal{N})$ respectively, depending on whether the spinors are real (\mathbb{R}), complex (\mathbb{C}) or quaternionic (\mathbb{H}) [1]. For $d = 3$ $\mathcal{N}_{max} = 16$, whereas for $d = 4$ and 5 $\mathcal{N}_{max} = 8$ (\mathcal{N} is only even for $d = 5$). In all cases the maximum number of (real) components of the spinor *supercharges* is $n_{max} = 32$ [2, 3].

Thus, \mathcal{N} -extended *supergravity* is unique *iff* $16 < n \leq 32$, while the uniqueness of the theory breaks down for $n \leq 16$. Nevertheless, for $8 < n \leq 16$ the *non-linear sigma models*, also containing the scalars from the additional matter multiplets coupled to the *supergravity* one, are still described by *symmetric spaces* of the form $\frac{G_M}{H_{\mathcal{R}} \otimes H_M}$, where H_M is a classical *compact* Lie group depending on the theory under consideration. Once again, the *non-compact* global isometry group G_M is uniquely fixed by the number of scalar fields and by the fact that G_M is a *non-compact, real form* of a *simple* (finite-dimensional) Lie group $G_{M,\mathbb{C}}$, whose *mcs* is $H_{\mathcal{R}} \otimes H_M$ [2, 3].

In all aforementioned cases, the signature of the coset manifold is (negatively) Euclidean, *i.e.* we are dealing with *Riemannian (globally) symmetric spaces* [4, 5].

The considered *supergravity* theories are *invariant* under G - (or G_M -) diffeomorphisms, as well as under general coordinate diffeomorphisms in space-time. Fermion fields are assigned to a suitable representation of $H_{\mathcal{R}} (\otimes H_M)$, while spin-1 vector fields are in a suitable representation of $G_{(M)}$. Among the treated cases $d = 3, 4, 5$, an exception is given by $d = 4$, in which case $G_{(M)}$ may mix electric and magnetic spin-1 field strengths' components, and the equations of motions - *but not the Lagrangian density* - are invariant under $G_{(M)}$. This phenomenon is nothing but the generalization [6] of the electric-magnetic duality of Maxwell equations, in which $G = SL(2, \mathbb{R}) \sim SO(2, 1) \sim SU(1, 1) \sim Spin(2, 1)$, with *mcs* = $U(1)$, the electric field and the magnetic field transforming as a real spinor (doublet) of G .

2 Classification of Irreducible Riemannian Globally Symmetric Spaces

Irreducible Riemannian globally symmetric spaces (of the type I and type III in Helgason's classification; see [4, 5]), denoted with the acronym IRGS in the treatment given below, are those *symmetric spaces* with (strictly) negative definite metric signature. They have the form $\frac{G}{H}$, where G is a *non-compact, real form* of a *simple* (finite-dimensional) Lie group $G_{\mathbb{C}}$, and H is its *mcs* (with *symmetric embedding*; H is also often referred to as the *stabilizer* of the coset). There are seven *classical* (infinite) sequences, as well as twelve *exceptional* isolated cases (in which $G_{\mathbb{C}}$ is an *exceptional* Lie group).

Furthermore, another class of *symmetric spaces* exists, with form $\frac{G_{\mathbb{C}}}{G_{\mathbb{R}}}$ [4], where $G_{\mathbb{C}}$ is any complex (*non-compact*) (*semi*-)*simple* Lie group regarded as a real group, and $G_{\mathbb{R}}$ is its *compact, real form* (*mcs* ($G_{\mathbb{C}}$) = $G_{\mathbb{R}}$). $\frac{G_{\mathbb{C}}}{G_{\mathbb{R}}}$ is a Riemann *symmetric space* with $\dim_{\mathbb{R}} = \dim_{\mathbb{R}}(G_{\mathbb{R}})$, and $\text{rank} = \text{rank}(G_{\mathbb{R}})$. A remarkable example of such a class of IRGS is provided by the manifold $\frac{SO(3,1)}{SO(3)}$, with $G_{\mathbb{R}} = SO(3) \sim SU(2)$ and $G_{\mathbb{C}} = SL(2, \mathbb{C}) \sim SO(3, 1)$ (see *e.g.* [4]). Such a space is not *quaternionic*, despite having $SU(2)$ as stabilizer; consistently, its *real* dimension is 3 (not a multiple of 4, as instead it holds for all

quaternionic manifolds; see below). On the other hand, as yielded by the treatment of Sect. 3, the unique example of such a class playing a role in *supergravity* theories is the IRGS $\frac{SL(3,\mathbb{C})}{SU(3)}$ ($SU(3) = mcs(SL(3,\mathbb{C}))$ [4, 5, 7]), which is both the real special *symmetric* vector multiplets' scalar manifold in $\mathcal{N} = 2, d = 5$ *supergravity* based on the *Jordan algebra of degree three* $J_3^{\mathbb{C}}$, and the non-BPS $Z \neq 0$ moduli space of the corresponding theory in $d = 4$, obtained by reduction along a *spacelike* direction (see Table 4).

Let us recall here that the *symmetric* nature of a coset (*i.e.* *homogeneous*) manifold can be defined in purely algebraic terms through the so-called *Cartan's decomposition* of the Lie algebra \mathfrak{g} of a Lie group G :

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}, \quad (2.1)$$

where \mathfrak{h} is the Lie algebra of a *compact* H subgroup of G , and \mathfrak{k} can be identified with the tangent space at the identity coset. The *homogeneous* space $\frac{G}{H}$ is *symmetric* iff the three following properties hold (see *e.g.* [4, 5, 7]):

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}; \quad [\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}; \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}. \quad (2.2)$$

The first property (from the left) holds by definition of subgroup. The second property holds in general in coset spaces, and it means that by the adjoint, \mathfrak{h} acts on \mathfrak{k} as a representation R with $\dim_{\mathbb{R}}(R) = \dim_{\mathbb{R}}(\frac{G}{H})$. The third property defines the *symmetry* of the space under consideration, since in general it simply holds that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{g}$.

All IRGS are *Einstein spaces* (see *e.g.* [8, 9] and Refs. therein), thus with *constant* (*negative*) scalar curvature.

Moreover, one can define the *rank* of an IRGS is defined as the maximal dimension (in \mathbb{R}) of a *flat* (*i.e.* with vanishing Riemann tensor), *totally geodesic* submanifold of the IRGS itself (see *e.g.* §6, page 209 of [4]).

In the following treatment *Kähler* [10], *special Kähler* [11]–[27], *real special* [12, 13, 27, 28] and *quaternionic* [12, 13], [29]–[36], [17, 18, 20, 37, 38] manifolds are denoted by K, SK, RS and H, respectively. The role played by such spaces in *supergravity* is outlined in Sect. 3.

Tables 1 and 2 respectively list the seven *classical* infinite sequences and the twelve *exceptional* isolated cases (see *e.g.* Table II of [4]). Some observations are listed below (other properties are given in, or can be inferred from, Tables 3–11):

- \mathbf{I}_2 is SK
- \mathbf{I}_3 is not H, despite having $SO(3) \sim SU(2)$ as stabilizer; consistently, its *real* dimension is 10 (not a multiple of 4, as instead it holds for all H manifolds)
- $\mathbf{III}_{2,q} = \mathbf{III}_{p,2}$ is both H and K (*quaternionic Kähler*). In particular, $\mathbf{III}_{2,1} = \mathbf{III}_{1,2}$ is both H and SK, with $\dim_{\mathbb{R}} = 4 \Leftrightarrow \dim_{\mathbb{H}} = 1$, and it is an example of *Einstein space with self-dual Weyl curvature* [30]
- $\mathbf{IV}_{2,3} = \mathbf{IV}_{3,2}$ is K, but not H, despite having $SO(3) \otimes SO(2) \sim SU(2) \otimes U(1)$ as stabilizer; consistently, its *real* dimension is 6 (not a multiple of 4)
- $\mathbf{IV}_{2,4} = \mathbf{IV}_{4,2}$ is both H and K (*quaternionic Kähler*)
- \mathbf{V}_2 is K, but not H, despite having $U(2)$ as stabilizer. Through the isomorphism $SO^*(4) \sim SU(2) \otimes SL(2, \mathbb{R})$ [4], it holds that $\mathbf{V}_2 \sim \frac{SU(1,1)}{U(1)}$, with *real* dimension 2 (not a multiple of 4)
- \mathbf{VI}_2 is K, but not H, despite having $U(2)$ as stabilizer. Through the isomorphism $SO(3,2) \sim Sp(4, \mathbb{R})$ [4], it holds that $\mathbf{IV}_{2,3} \sim \mathbf{VI}_2$
- $\mathbf{VII}_{1,q} = \mathbf{VII}_{p,1} \equiv \mathbb{H}P^q = \mathbb{H}P^p$ (*quaternionic projective sequence*) is H, and it is the *unique symmetric* H space which is *not* the *c-map* of a *symmetric* SK space [59] (see Table 3)
- When the stabilizer of \mathbf{VIII}_G contains an explicit $U(1)$ factor, then \mathbf{VIII}_G *may be* (but in general *not necessarily is*) K
- When the stabilizer of \mathbf{VIII}_G contains an explicit $SU(2)$ factor, then \mathbf{VIII}_G *may be* (but in general *not necessarily is*) H

3 Irreducible Riemannian Globally Symmetric Spaces in Supergravity

Supergravity is a theory which combines general covariance (*diffeomorphisms*) with *local* supersymmetry (*superdiffeomorphisms*). It contains a *tetrad* (*Vielbein*) one-form e^a and a *gravitino* (spinor valued) one-form ψ_A^α ($a = 1, \dots, \mathcal{N}$), which for instance appear in the *Einstein-Hilbert* Lagrangian $\epsilon R \wedge e \wedge e$ (ϵ and R respectively being the Levi-Civita and Riemann tensors), or in the *Rarita-Schwinger* Lagrangian $\bar{\psi} \wedge d\psi \wedge \gamma$ (γ denoting the appropriate set of *gamma matrices*). The Lagrangians of the gauge fields are of the form $(\text{Re}\mathcal{N}_{\Lambda\Sigma}) F^\Lambda \wedge F^\Sigma$ and $(\text{Im}\mathcal{N}_{\Lambda\Sigma}) F^\Lambda \wedge *F^\Sigma$, where $\mathcal{N}_{\Lambda\Sigma}$ is a complex symmetric *kinetic vector matrix*.

Symmetric spaces already occurs in gravity, regardless of supersymmetry. A simple example is provided by the Kaluza-Klein reduction of D -dimensional gravity on a manifold

$$M_D = M_d \otimes M_{D-d}, \quad (3.1)$$

where the *internal* manifold is here taken to be a d -dim. *torus* (*i.e.* $M_d = T^d$) for simplicity's sake. For small size of T^d , the Kaluza-Klein reduction of *pure* gravity as given by Eq. (3.1) yields $(D - d)$ -dim. gravity coupled to $\frac{d(d+1)}{2}$ scalar fields and d Maxwell fields (*graviphotons*). The scalar fields parameterize (as coordinates) the manifold $\frac{GL(d, \mathbb{R})}{SO(d)}$; modding out the overall size of T^d , one obtains the IRGS $\frac{SL(d, \mathbb{R})}{SO(d)}$ (see Table I), which is the simplest example of *symmetric* space occurring in gravity.

Supersymmetry restricts the *holonomy* group of Riemannian spaces which may occur in a given theory (see *e.g.* [2, 3]). Let us consider for instance *supergravity* theories in $d = 4$ space-time dimensions. The geometry of the scalar manifolds depends on the number \mathcal{N} of *supercharges*: it is K [10] for $\mathcal{N} = 1$, SK [11]–[27] (for vector multiplets' scalars) or H [12, 13], [29]–[36], [17, 18, 20, 37, 38] (for hypermultiplets' scalars) for $\mathcal{N} = 2$, and in general *symmetric* for $\mathcal{N} > 2$. Concerning $\mathcal{N} = 2$ *supergravity* in $d = 5$ and $d = 3$ space-time dimensions, the vector multiplets' scalar manifolds are endowed with RS [12, 13, 27, 28] and H [12, 13], [29]–[36], [17, 18, 20, 37, 38] geometry, respectively. The isolated cases of *symmetric* SK manifolds are given by the so-called *magic* $\mathcal{N} = 2$ *supergravities* ([39, 40], see Table 3). They are related to *Freudenthal triple systems* [40]–[46] over the simple Euclidean *Jordan algebras* [39, 40], [47]–[52] of degree three with irreducible norm forms, namely over the *Jordan algebras* $J_3^\mathbb{O}$, $J_3^\mathbb{H}$, $J_3^\mathbb{C}$ and $J_3^\mathbb{R}$ of Hermitian 3×3 matrices over the four *division algebras*, *i.e.* respectively over the *octonions* (\mathbb{O}), *quaternions* (\mathbb{H}), *complex numbers* (\mathbb{C}) and *real numbers* (\mathbb{R}). Furthermore, they are also connected to the *Magic Square* of Freudenthal, Rozenfeld and Tits [41, 53, 54, 40, 39] (see also, for recent treatment, [55]–[58]). *Jordan algebras* were introduced and completely classified in [49] in an attempt to generalize *Quantum Mechanics* beyond the field of complex numbers \mathbb{C} .

The scalar manifolds of $\mathcal{N} > 2$ *pure supergravities* in $d = 3, 4, 5$ are all *symmetric*, of the form $\frac{G_{d, \mathcal{N}}}{H_{d, \mathcal{N}}}$, where, as anticipated in the Introduction, $H_{d, \mathcal{N}}$ is nothing but the automorphism group of the related \mathcal{N} -extended, d -dim. *superalgebra*, usually named \mathcal{R} -symmetry group. As mentioned in the Introduction, in $d = 3, 4$ and 5 the \mathcal{R} -symmetry is $SO(\mathcal{N})$, $U(\mathcal{N})$ and $USp(\mathcal{N})$ respectively, depending on whether the spinors are *real*, *complex* or *quaternionic* (see *e.g.* Table 2 of [1]). Since from *group representation theory* the number of scalar fields in the corresponding *supergravity* multiplet is known (being related to the relevant *Clifford algebra* - see *e.g.* [1] -), the global isometry group $G_{d, \mathcal{N}}$ is determined uniquely, at least locally.

A set of Tables shows the role played by IRGS in *supergravities* with \mathcal{N} *supercharges* in $d = 3, 4, 5$ space-time dimensions.

- Table 3 presents the relation among $\mathcal{N} = 2$, $d = 4$ *symmetric* SK vector multiplets' scalar manifolds and the *symmetric* H scalar manifolds of the corresponding $d = 3$ theory obtained by *spacelike* dimensional reduction (or equivalently of the $d = 4$ hypermultiplets' scalar manifolds), given by the so-called *c-map* [59]. The *c-map* of *symmetric* SK manifolds gives the whole set of *symmetric* H manifolds, the unique exception being the *quaternionic projective spaces* $\mathbb{H}P^n$ introduced above: they are *symmetric* H manifolds which are *not* the *c-map* of any (*symmetric*) SK space¹. Furthermore, all *symmetric* SK manifolds but the *complex projective spaces* $\mathbb{C}P^n$ (and thus, through *c-map*,

¹Many other H manifolds exist, such as the *homogeneous non-symmetric* ones studied in [34] and the (rather general, *not necessarily homogeneous*) ones given by the *c-map* of general SK geometries (they are *not* completely general, because they are endowed with $2n + 4$ isometries, if the corresponding SK geometry has $\dim_{\mathbb{C}} = n$) [36]. All H manifolds are *Einstein*, with *constant (negative)* scalar curvature (see *e.g.* [37, 38]).

all *symmetric* H manifolds but $\mathbb{H}\mathbb{P}^n$) are related to a *Jordan algebra of degree three*. In Table 3 \mathbb{R} denotes the one-dimensional *Jordan algebra*, whereas $\mathbf{\Gamma}_{m,n}$ stands for the *Jordan algebra of degree two* with a quadratic form of Lorentzian signature (m, n) , which is nothing but the *Clifford algebra* of $O(m, n)$ [49]. Furthermore, it is here worth pointing out that the theory with 8 supersymmetries based on the Jordan algebra $J_3^{\mathbb{H}}$ is *dual* to the *supergravity* with 24 supersymmetries, in $d = 3, 4, 5$ dimensions: they share the same scalar manifold, and the same number (and representation) of vector fields (see *e.g.* [56, 75], and Refs. therein)

- Table 4 lists the *moduli spaces* associated to *non-degenerate* non-BPS $Z \neq 0$ extremal black hole *attractors* in $\mathcal{N} = 2, d = 4$ SK *symmetric* vector multiplets' scalar manifolds [60]. They are nothing but the $\mathcal{N} = 2, d = 5$ RS *symmetric* vector multiplets' scalar manifolds. Only another class of $\mathcal{N} = 2, d = 5$ RS *symmetric* vector multiplets' scalar manifolds exists, namely the infinite sequence $\mathbf{IV}_{1,n-1} = \frac{SO(1,n-1)}{SO(n-1)}$, $n \in \mathbb{N}$, usually denoted by $L(-1, n-2)$ in the classification of *homogeneous d-spaces* [12]. It corresponds to *homogeneous non-symmetric* scalar manifolds in $d = 4$ (SK) and 3 (H) space-time dimensions (see *e.g.* Table 2 of [12]).

In general, an extremal black hole *attractor* is associated to a (stable) critical point of a suitably defined *black hole effective potential* V_{BH} , and it describes a scalar configuration, stabilized *purely* in terms of the conserved electric and magnetic charges at the *event horizon*, *regardless* of the values of the scalars at spatial infinity. This is due to the *Attractor Mechanism* [61]–[64], an important dynamical phenomenon in the theory of gravitational objects, which naturally appears in modern theories of gravity, such as *supergravity*, *superstrings* [65]–[68] or *M-theory* [69]–[71].

In *homogeneous* (not necessarily *symmetric*) scalar manifolds $\frac{G}{H}$, the horizon *attractor* configurations of the scalar fields are supported by *non-degenerate orbits* (*i.e.* orbits with *non-vanishing* classical entropy) of the representation of the charge vector in the group G , which can thus be used in order to classify the various typologies of *attractors*. A complete classification of the (*non-degenerate*) *charge orbits* \mathcal{O} has been performed for all *supergravities* based on *symmetric* scalar manifolds in $d = 4$ and 5 dimensions [44, 56, 60], [72]–[79]. In such a framework, the *charge orbits* \mathcal{O} are *homogeneous* (generally *non-symmetric*) manifolds (with Lorentzian signature) of the form $\frac{G}{\mathfrak{H}}$, where \mathfrak{H} is some proper subgroup of G . If \mathfrak{H} is *non-compact*, then a *moduli space* can be associated to the *charge orbit* (and thus to the corresponding class of *attractors*): it is an IRGS of the form $\frac{\mathfrak{H}}{H}$, where $H = mcs(\mathfrak{H})$ (with *symmetric* embedding) [60, 78, 79]. The *moduli space* $\frac{\mathfrak{H}}{H}$ is spanned by those scalar degrees of freedom which are *not* stabilized in terms of charges at the *event horizon* of the considered extremal black hole. In other words, $\frac{\mathfrak{H}}{H}$ describes the *flat directions* of the relevant V_{BH} at the considered class of *non-degenerate* attractors. Within such a framework, the fact that in $\mathcal{N} = 2, d = 4, 5$ *supergravity* the $\frac{1}{2}$ -BPS *attractors* stabilize *all* scalars at the event horizon can be traced back, in the case of *symmetric* vector multiplets' scalar manifold, to the *compactness* of the stabilizer $H_{\frac{1}{2}-BPS}$ of the corresponding $\frac{1}{2}$ -BPS supporting *charge orbit* $\mathcal{O}_{\frac{1}{2}-BPS} = \frac{G}{H_{\frac{1}{2}-BPS}}$.

Recent studies [80]–[84] suggest that the moduli spaces of *non-degenerate* attractors do *not* exist only at the event horizon of the considered extremal black hole, but rather they can be extended (with no changes) *all along the corresponding attractor flow*, *i.e.* all along the evolution dynamics of the scalar fields (determined by the scalar equations of motion), from the *spatial infinity* $r \rightarrow \infty$ to the *near-horizon geometry* ($r \rightarrow r_H^+$), r and r_H being the radial coordinate and the radius of the event horizon, respectively. However, such *moduli spaces* are *not* expected to survive the *quantum corrections* to the classical geometry of the scalar manifolds, as confirmed (*at least* in some black hole charge configurations) in [85].

Turning back to Table 4, \hat{H} denotes the *non-compact* stabilizer of the corresponding supporting *charge orbit* $\mathcal{O}_{non-BPS, Z \neq 0}$ [74], and \hat{h} is its *mcs* (with *symmetric* embedding)

- Table 5 presents the *moduli spaces* of non-BPS $Z = 0$ critical points of $V_{BH, \mathcal{N}=2}$ in $\mathcal{N} = 2, d = 4$ SK *symmetric* vector multiplets' scalar manifolds [60]. They are (non-special) Kähler *symmetric* manifolds. \hat{H} denotes the *non-compact* stabilizer of the corresponding supporting *charge orbit* $\mathcal{O}_{non-BPS, Z=0}$ [74], and \hat{h} is its *mcs* (with *symmetric* embedding). Remarkably, $\frac{E_{6(-14)}}{SO(10) \otimes U(1)}$ is associated to $M_{1,2}(\mathbb{O})$, which is another exceptional *Jordan triple system*, generated by 2×1

Hermitian matrices over the octonions \mathbb{O} , found in [40, 39]. Furthermore, $\frac{E_{6(-14)}}{SO(10) \otimes U(1)}$ is also the scalar manifold of $\mathcal{N} = 10$, $d = 3$ *supergravity* (see Table 11 below, and Table 2 of [55], as well)

- Table 6 contains the scalar manifolds of $\mathcal{N} \geq 3$ -extended, $d = 4$ *supergravities*. $J_3^{\mathbb{O}_s}$ denotes the *Jordan algebra of degree three* over the *split form* \mathbb{O}_s of the octonions (see e.g. [86] and Refs. therein for further, and recent, developments). Remarkably, $M_{1,2}(\mathbb{O})$ is also associated to $\mathcal{N} = 5$, $d = 4$ *supergravity* (see Table 2 of [55], and Refs. therein)
- Table 7 lists the *moduli spaces* of *non-degenerate* extremal black hole *attractors* in $3 \leq \mathcal{N} \leq 8$, $d = 4$ *supergravities* [60, 87], [76]–[78]. \mathfrak{h} , $\hat{\mathfrak{h}}$ and $\tilde{\mathfrak{h}}$ respectively are the *mcs*'s (with *symmetric* embedding) of \mathcal{H} , $\hat{\mathcal{H}}$ and $\tilde{\mathcal{H}}$, which in turn are the *non-compact* stabilizers of the corresponding supporting charge orbits $\mathcal{O}_{1/\mathcal{N}-BPS}$, $\mathcal{O}_{non-BPS, Z_{AB} \neq 0}$ and $\mathcal{O}_{non-BPS, Z_{AB} = 0}$, respectively [44, 56, 60], [74]–[78] (see Table 1 of [78]). It is here worth recalling that *all non-degenerate* $\frac{1}{\mathcal{N}}$ -BPS *moduli spaces* $\frac{\mathcal{H}}{\mathfrak{h}}$ (see Table 7) and $\frac{\tilde{\mathcal{H}}_5}{\tilde{\mathfrak{h}}_5}$ (see Table 10) of $8 \geq \mathcal{N} > 2$ -extended *supergravities* in $d = 4, 5$ space-time dimensions are H manifolds. This has a nice interpretation in terms of $\mathcal{N} \longrightarrow 2$ *supersymmetry reduction*: the *flat directions* of $V_{BH, \mathcal{N}}$ at the considered class of its (*non-degenerate*) critical points correspond to the would-be hypermultiplets' scalar degrees of freedom in the *vector/hyper splitting* determined by the $\mathcal{N} \longrightarrow 2$ *supersymmetry reduction* [87]–[89], [77, 60, 76]
- Table 8 shows the *moduli spaces* of *non-degenerate* non-BPS ($Z \neq 0$) critical points of $V_{BH, \mathcal{N}=2}$ in $\mathcal{N} = 2$, $d = 5$ RS *symmetric* vector multiplets' scalar manifolds [60]. \tilde{H}_5 stands for the *non-compact* stabilizer of the corresponding supporting charge orbit $\mathcal{O}_{non-BPS}$ [60], and \tilde{K}_5 is its *mcs* (with *symmetric* embedding)
- Table 9 lists the scalar manifolds of $\mathcal{N} > 2$ -extended, $d = 5$ *supergravities*
- Table 10 presents the *moduli spaces* of extremal black hole attractors with *non-vanishing* classical entropy in $4 \leq \mathcal{N} \leq 8$ -extended, $d = 5$ *supergravities* [77, 60, 79]. \mathfrak{h}_5 and $\hat{\mathfrak{h}}_5$ respectively are the *mcs*'s (with *symmetric* embedding) of \mathcal{H}_5 and $\hat{\mathcal{H}}_5$, which in turn are the *non-compact* stabilizers of the corresponding supporting charge orbits $\mathcal{O}_{1/\mathcal{N}-BPS}$ and $\mathcal{O}_{non-BPS}$, respectively [44, 75, 56, 77, 60, 79]
- Finally, Table 11 contains the scalar manifolds of $\mathcal{N} \geq 5$, $d = 3$ *supergravities* [29].

As yielded by Tables 3-11, *all* typologies of IRGS appear *at least* once in *supergravity* theories with \mathcal{N} *supercharges* in $d = 3, 4, 5$ space-time dimensions (as scalar manifolds, or as *moduli spaces* associated to the various classes of extremal black hole attractors with *non-vanishing* classical entropy).

Let us now consider the *supergravities* with 8 supersymmetries associated to the *Jordan algebras of degree three* $J_3^{\mathbb{A}}$ over the four *division algebras* $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and \mathbb{O} , shortly called *magic supergravities*, in $d = 3, 4$ and 5 space-time dimensions. By recalling the Tables 3,4,5 and 8 and recalling the definition $A \equiv \dim_{\mathbb{R}}(\mathbb{A}) = 1, 2, 4, 8$ (for $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and \mathbb{O} respectively) (see Table 3), one gets that [90]

$$\dim_d \mathcal{M}_{d, J_3^{\mathbb{A}}} = 3A + 7 - d; \quad (3.2)$$

$$\dim_d \mathcal{F}_{d, J_3^{\mathbb{A}}} = 2A; \quad (3.3)$$

$$\dim_d \equiv \dim_{\mathbb{R}} \Big|_{d=5}, \dim_{\mathbb{C}} \Big|_{d=4}, \dim_{\mathbb{H}} \Big|_{d=3}. \quad (3.4)$$

In Eq. (3.2) $\mathcal{M}_{d, J_3^{\mathbb{A}}}$ denotes the scalar manifold of the *supergravity* theory with 8 supersymmetries associated to $J_3^{\mathbb{A}}$ in $d (= 3, 4, 5)$ space-time dimensions. In Eq. (3.3) $\mathcal{F}_{d, J_3^{\mathbb{A}}}$ stands for the set of non-BPS $Z = 0$ *moduli spaces* of *symmetric* $J_3^{\mathbb{A}}$ -related SK manifolds (see Table 5), and $\mathcal{F}_{5, J_3^{\mathbb{A}}}$ is the set of non-BPS ($Z \neq 0$) *moduli spaces* of *symmetric* $J_3^{\mathbb{A}}$ -related RS manifolds (see Table 8). Let us now consider the finite sequence (for $A = 1, 2, 4, 8$) of $(\mathbb{R} \oplus \Gamma_{A+1,1})$ -related *symmetric* $d = 4$ SK manifolds $\mathbf{III}_{1,1} \otimes \mathbf{IV}_{2, A+2} \equiv \mathcal{B}_{4, A}$ (Table 3), as well as its *c-map* sequence $\mathbf{IV}_{4, A+4} \equiv \mathcal{B}_{3, A}$ (Table 3) and the

corresponding (through a $d = 5 \rightarrow 4$ *dimensional reduction* along a *spacelike* direction) sequence of RS *symmetric* spaces $SO(1, 1) \otimes \mathbf{IV}_{1,A+1} \equiv \mathcal{B}_{5,A}$ (Table 4):

$$\begin{aligned}
\mathcal{B}_{5,A} &\equiv SO(1, 1) \otimes \mathbf{IV}_{1,A+1} : SO(1, 1) \otimes \frac{SO(1, A+1)}{SO(A+1)}, \dim_{\mathbb{R}} = A+2; \\
&\quad \downarrow \\
\mathcal{B}_{4,A} &\equiv \mathbf{III}_{1,1} \otimes \mathbf{IV}_{2,A+2} : \frac{SU(1, 1)}{U(1)} \otimes \frac{SO(2, A+2)}{SO(A+2) \otimes U(1)}, \dim_{\mathbb{C}} = A+3; \\
&\quad \downarrow c\text{-map} \\
\mathcal{B}_{3,A} &\equiv \mathbf{IV}_{4,A+4} : \frac{SO(4, A+4)}{SO(A+4) \otimes SO(4)}, \dim_{\mathbb{H}} = A+4.
\end{aligned} \tag{3.5}$$

It is thus evident that

$$\dim_d \mathcal{B}_{d,A} = A+7-d = \dim_d \mathcal{M}_{d,J_3^A} - \dim_d \mathcal{F}_{d,J_3^A}. \tag{3.6}$$

Actually, as found in [90], \mathcal{M}_{d,J_3^A} has a non-trivial *bundle structure*, where the manifold \mathcal{F}_{d,J_3^A} is *fibered* over the *base manifold* $\mathcal{B}_{d,A}$:

$$\mathcal{M}_{d,J_3^A} = \mathcal{B}_{d,A} + \mathcal{F}_{d,J_3^A}. \tag{3.7}$$

The four elements of the finite sequence \mathcal{F}_{3,J_3^A} are uniquely determined by requiring that $\mathcal{F}_{3,J_3^A} \subset \mathcal{M}_{3,J_3^A}$ and that $\dim_{\mathbb{H}} \mathcal{F}_{3,J_3^A} = 2A$ [90]. Notice that in general

$$\mathcal{M}_{3,J_3^A} = c\text{-map} \left(\mathcal{M}_{4,J_3^A} \right), \quad \mathcal{B}_{3,A} = c\text{-map} \left(\mathcal{B}_{4,A} \right), \tag{3.8}$$

but

$$\mathcal{F}_{3,J_3^A} \neq c\text{-map} \left(\mathcal{F}_{4,J_3^A} \right), \tag{3.9}$$

and analogously it holds for the relation between $d = 5$ and $d = 4$ space-time dimensions. For example (see Table 3 [59])

$$\mathcal{F}_{3,J_3^0} = \frac{E_{7(-5)}}{SO(12) \otimes SU(2)} = c\text{-map} \left(\frac{SO^*(12)}{SU(6) \otimes U(1)} \right), \tag{3.10}$$

and (see *e.g.* [5])

$$\frac{SO^*(12)}{SU(6) \otimes U(1)} \not\supset \not\subset \mathcal{F}_{4,J_3^0} \left(= \frac{E_{6(-14)}}{SO(10) \otimes U(1)} \right); \tag{3.11}$$

$$\frac{SO^*(12)}{SU(6) \otimes U(1)} \cap \frac{E_{6(-14)}}{SO(10) \otimes U(1)} = \frac{SO^*(10)}{SU(5) \otimes U(1)} = \mathbf{V}_5. \tag{3.12}$$

Concerning the stringy interpretation(s) of the *fiber bundle decomposition* (3.7) of \mathcal{M}_{d,J_3^A} , in (Type I) string theory the *base* $\mathcal{B}_{d,A}$ should describe *closed string moduli*, while the *fiber* \mathcal{F}_{d,J_3^A} describes *open string moduli*.

Thus, one obtains twelve *fiber bundle decompositions* of J_3^A -related *supergravity* models, forming three *sequences* of four *exceptional* geometries. Tables 12, 13 and 14 list such *exceptional sequences* in $d = 5$, 4 and 3 space-time dimensions, respectively [90]. It is worth noticing that $\mathcal{B}_{4,8}$ is nothing but the vector multiplets' scalar manifold of the so-called *FHSV model* [91], studied in [92]–[96], and correspondingly $\mathcal{B}_{5,8}$ and $\mathcal{B}_{3,8}$ respectively are its $d = 5$ *uplift* and its *c-map*. The sequence $\left\{ \mathcal{F}_{4,J_3^A} \right\}_{A=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}}$, given by the fourth column of $d = 4$ *exceptional sequence* (Table 13) has also been recently found in a framework which connects *magic supergravities* to *constrained instantons* [58]. The other two sequences $\left\{ \mathcal{F}_{5,J_3^A} \right\}_{A=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}}$ and $\left\{ \mathcal{F}_{3,J_3^A} \right\}_{A=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}}$, respectively given by the fourth column of $d = 5$ and $d = 3$ *exceptional sequences* (Tables 12 and 14, respectively), are new to our knowledge.

It is interesting to notice that Kostant, through a construction based on *minimal coadjoint orbits* and *symplectic induction* [97], related *Jordan algebras of degree four* to IRGS $\frac{G}{K}$, in which G is a particular

non-compact real form of a simple *exceptional* (finite-dimensional) Lie group, and K is its (*symmetrically* embedded) *mcs*. The IRGS $\frac{G}{K}$ appearing in Kostant's construction (summarized by Table in page 422 of [97], reported below in Table 15) are two H manifolds, which are the *c-map* of the so-called t^3 model ($G = G_{2(2)}$) and of the *real magic* $\mathcal{N} = 2$, $d = 4$ *supergravity* ($G = F_{4(4)}$) [59], respectively based on the *Jordan algebras* \mathbb{R} (*degree one*) and $J_3^{\mathbb{R}}$ (*degree three*), as well as the scalar manifolds of *maximal supergravity* in $d = 3, 4, 5$ space-time dimensions ($G = E_{8(8)}, E_{7(7)}, E_{6(6)}$ respectively), based on $J_3^{\mathbb{O}^*}$. Through *symplectic induction* [97], they are connected to some *compact symmetric* Kähler spaces $X = \frac{K}{H_K}$, H_K being some proper (*symmetrically* embedded) *compact* subgroup of K . X is related to a *Jordan algebra* $J(X)$, with $\dim_{\mathbb{R}}(X) = 2\dim_{\mathbb{R}}(J(X))$. For $G = G_{2(2)}$, this is a *Jordan algebra of degree two*, whereas in all other cases it has *degree four*. Consistently with previous notation, in Table 15 $J_4^{\mathbb{R}}, J_4^{\mathbb{C}}, J_4^{\mathbb{H}}$ respectively denote the *Jordan algebras of degree four* with irreducible norm forms, made by Hermitian 4×4 matrices over \mathbb{R}, \mathbb{C} and \mathbb{H} . It is worth remarking here that X has an associated (still Kähler) *symmetric non-compact* form $\mathcal{X} = \frac{K}{H_K}$, which is an (I)RGS, with $K \subset G$. Furthermore, \mathcal{X} is *unique*, because *only one non-compact, real* form K of K exists, such that $K \subset G$ and $mcs(K) = H_K$ (see e.g. [5]). Notice also that $rank(X) = rank(\mathcal{X})$ is also the *degree* of the corresponding $J(X)$. It is amusing to observe that $\dim_{\mathbb{R}}(X)$ is also the *real* dimension of the representation R_V of the Abelian vector field strengths (and of their dual) in $\mathcal{N} = 2$, $d = 4$ *magic supergravities* over $\mathbb{O}, \mathbb{H}, \mathbb{C}$ and \mathbb{R} , as well as of the so-called t^3 model [40, 39, 44, 74]. It would be interesting to study further such a construction, and determine the origin of the (I)RGS \mathcal{X} in *supergravity*.

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References

- [1] J. A. Strathdee, *Extended Poincaré Supersymmetry*, Int. J. Mod. Phys. **A2**, 273 (1987); P. Deligne, *Notes on Spinors*, in : “*Quantum Fields and Strings : a Course for Mathematicians*” (American Mathematical Society, Providence, 1999); R. D'Auria, S. Ferrara, M. A. Lledó and V. S. Varadarajan, *Spinor Algebras*, J. Geom. Phys. **40**, 101 (2001), [hep-th/0010124](#).
- [2] S. Ferrara, J. Scherk and B. Zumino, *Supergravity and Local Extended Supersymmetry*, Phys. Lett. **B66**, 35 (1977).
- [3] S. Ferrara, J. Scherk and B. Zumino, *Algebraic Properties of Extended Supergravity Theories*, Nucl. Phys. **B121**, 393 (1977).
- [4] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces* (Academic Press, New York, 1978).
- [5] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (Dover Publications, 2006).
- [6] M. K. Gaillard and B. Zumino, *Duality Rotations for Interacting Fields*, Nucl. Phys. **B193**, 221 (1981).
- [7] R. Slansky, *Group Theory for Unified Model Building*, Phys. Rep. **79**, 1 (1981).

- [8] L. Castellani, R. D'Auria, P. Fré, “*Supergravity and superstrings: A Geometric perspective*”, 3 Vols. (World Scientific, Singapore), 1991.
- [9] P. Fré and P. Soriani, “*The $\mathcal{N}=2$ wonderland: From Calabi-Yau manifolds to topological field theories*” (World Scientific, Singapore), 1995.
- [10] B. Zumino, *Supersymmetry and Kähler Manifolds*, Phys. Lett. **B87**, 203 (1979).
- [11] E. Cremmer and A. Van Proeyen, *Classification of Kähler Manifolds in $\mathcal{N}=2$ Vector Multiplet Supergravity Couplings*, Class. Quant. Grav. **2**, 445 (1985).
- [12] B. de Wit, F. Vanderseypen and A. Van Proeyen, *Symmetry Structures of Special Geometries*, Nucl. Phys. **B400**, 463 (1993), [hep-th/9210068](#).
- [13] B. de Wit and A. Van Proeyen, *Special geometry, cubic polynomials and homogeneous quaternionic spaces*, Commun. Math. Phys. **149**, 307 (1992), [hep-th/9112027](#).
- [14] S. Cecotti, *$\mathcal{N}=2$ Supergravity, Type IIB Superstrings And Algebraic Geometry*, Commun. Math. Phys. **131**, 517 (1990).
- [15] L. Castellani, R. D'Auria and S. Ferrara, *Special Kähler Geometry: An Intrinsic Formulation From $\mathcal{N}=2$ Space-Time Supersymmetry*, Phys. Lett. **B241**, 57 (1990).
- [16] L. Castellani, R. D'Auria and S. Ferrara, *Special geometry without special coordinates*, Class. Quant. Grav. **7**, 1767 (1990).
- [17] R. D'Auria, S. Ferrara and P. Fré, *Special and quaternionic isometries: General couplings in $\mathcal{N}=2$ supergravity and the scalar potential*, Nucl. Phys. **B359**, 705 (1991).
- [18] B. de Wit and A. Van Proeyen, *Hidden symmetries, special geometry and quaternionic manifolds*, Int. J. Mod. Phys. **D3**, 31 (1994), [hep-th/9310067](#).
- [19] B. de Wit and A. Van Proeyen, *Special geometry and symplectic transformations*, Nucl. Phys. Proc. Suppl. **45BC**, 196 (1996), [hep-th/9510186](#).
- [20] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré, T. Magri, *$\mathcal{N}=2$ supergravity and $N=2$ superYang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map*, J. Geom. Phys. **23**, 111 (1997), [hep-th/9605032](#).
- [21] Daniel S. Freed, *Special Kähler manifolds*, Commun. Math. Phys. **203**, 31 (1999), [hep-th/9712042](#).
- [22] B. Craps, F. Roose, W. Troost and A. Van Proeyen, *What is special Kähler geometry?*, Nucl. Phys. **B503**, 565 (1997), [hep-th/9703082](#).
- [23] P. Fré, *Lectures on special Kähler geometry and electric - magnetic duality rotations*, Nucl. Phys. Proc. Suppl. **45BC**, 59 (1996), [hep-th/9512043](#).
- [24] Z. Lu, *A note on special Kähler manifolds*, Math. Ann. **313**, 711 (1999).
- [25] A. Strominger, *Special Geometry*, Commun. Math. Phys. **133**, 163 (1990).
- [26] B. de Wit and A. Van Proeyen, *Potentials and Symmetries of General Gauged $\mathcal{N}=2$ Supergravity: Yang-Mills Models*, Nucl. Phys. **B245**, 89 (1984).
- [27] M. Lledó, *Special geometry of $d=4,5$ supersymmetry*, talk given at the Conference on *Symmetry in Mathematics & Physics*, IPAM, UCLA, Los Angeles, 18-20 Jan. 2008.
- [28] B. de Wit and A. Van Proeyen, *Broken sigma model isometries in very special geometry*, Phys. Lett. **B293**, 94 (1992), [hep-th/9207091](#).
- [29] B. de Wit, A. K. Tollstén and H. Nicolai, *Locally supersymmetric $D=3$ nonlinear sigma models*, Nucl. Phys. **B392**, 3 (1993), [hep-th/9208074](#).
- [30] J. Bagger and E. Witten, *Matter Couplings in $\mathcal{N}=2$ Supergravity*, Nucl. Phys. **B222**, 1 (1983).

- [31] C. K. Zachos, *$\mathcal{N}=2$ Supergravity Theory With A Gauged Central Charge*, Phys. Lett. **B76**, 329 (1978).
- [32] P. Breitenlohner and M. F. Sohnius, *Superfields, Auxiliary Fields, And Tensor Calculus For $\mathcal{N}=2$ Extended Supergravity*, Nucl. Phys. **B165**, 483 (1980).
- [33] S. Salamon, Invent. Math. **67**, 143 (1982).
- [34] J. A. Wolf, J. Math. Mech. **14**, 1033 (1965).
- [35] D. V. Alekseevskii, Math. USSR Izv. **9**, 297 (1975).
- [36] S. Ferrara and S. Sabharwal, *Quaternionic Manifolds for Type II Superstring Vacua of Calabi-Yau Spaces*, Nucl. Phys. **B332**, 317 (1990).
- [37] S. Ishihara, *Quaternion Kählerian manifolds*, J. Differential Geom. **9**, 483 (1974).
- [38] F. Gürsey and H. C. Tze, *Complex And Quaternionic Analyticity In Chiral And Gauge Theories. Part 1*, Annals Phys. **128**, 29 (1980).
- [39] M. Günaydin, G. Sierra and P. K. Townsend, *The Geometry of $\mathcal{N}=2$ Maxwell-Einstein Supergravity and Jordan Algebras*, Nucl. Phys. **B242**, 244 (1984).
- [40] M. Günaydin, G. Sierra and P. K. Townsend, *Exceptional Supergravity Theories and the Magic Square*, Phys. Lett. **B133**, 72 (1983).
- [41] H. Freudenthal, Proc. Konink. Ned. Akad. Wetenschap **A62**, 447 (1959).
- [42] H. Freudenthal, Adv. Math. **1**, 145 (1964).
- [43] M. Günaydin, K. Koepsell and H. Nicolai, *Conformal and quasiconformal realizations of exceptional Lie groups*, Commun. Math. Phys. **221**, 57 (2001), [hep-th/0008063](#).
- [44] S. Ferrara and M. Günaydin, *Orbits of Exceptional Groups, Duality and BPS States in String Theory*, Int. J. Mod. Phys. **A13**, 2075 (1998), [hep-th/9708025](#).
- [45] M. Günaydin, *Unitary realizations of U-duality groups as conformal and quasiconformal groups and extremal black holes of supergravity theories*, AIP Conf. Proc. **767**, 268 (2005), [hep-th/0502235](#).
- [46] M. Günaydin and O. Pavlyk, *Generalized spacetimes defined by cubic forms and the minimal unitary realizations of their quasiconformal groups*, JHEP **0508**, 101 (2005), [hep-th/0506010](#).
- [47] M. Günaydin, G. Sierra and P. K. Townsend, *Gauging the $d=5$ Maxwell-Einstein Supergravity Theories: More on Jordan Algebras*, Nucl. Phys. **B253**, 573 (1985).
- [48] M. Günaydin, G. Sierra and P. K. Townsend, *More on $d=5$ Maxwell-Einstein Supergravity: Symmetric Space and Kinks*, Class. Quant. Grav. **3**, 763 (1986).
- [49] P. Jordan, J. Von Neumann and E. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, Ann. Math. **35**, 29 (1934).
- [50] N. Jacobson, Ann. Math. Soc. Coll. Publ. **39** (1968).
- [51] M. Günaydin, *Exceptional Realizations of Lorentz Group: Supersymmetries and Leptons*, Nuovo Cimento **A29**, 467 (1975).
- [52] M. Günaydin, C. Piron and H. Ruegg, *Moufang Plane and Octonionic Quantum Mechanics*, Comm. Math. Phys. **61**, 69 (1978).
- [53] B. A. Rozenfeld, Dokl. Akad. Nauk. SSSR **106**, 600 (1956).
- [54] J. Tits, Mem. Acad. Roy. Belg. Sci. **29**, fasc. 3 (1955).

- [55] B. Pioline, *Lectures on black holes, topological strings and quantum attractors*, Lectures delivered at the *RTN Winter School on Strings, Supergravity and Gauge Theories*, Geneva, Switzerland, 16-20 Jan 2006, *Class. Quant. Grav.* **23**, S981 (2006), [hep-th/0607227](#).
- [56] S. Ferrara, E. G. Gimon and R. Kallosh, *Magic supergravities, $\mathcal{N}=8$ and black hole composites*, *Phys. Rev.* **D74**, 125018 (2006), [hep-th/0606211](#).
- [57] M. Rios, *Jordan Algebras and Extremal Black Holes*, based on talk given at the *26th International Colloquium on Group Theoretical Methods in Physics (ICGTMP26)*, New York, 26-30 June 2006, [hep-th/0703238](#).
- [58] K. Dasgupta, V. Hussin and A. Wissanji, *Quaternionic Kahler Manifolds, Constrained Instantons and the Magic Square. I*, *Nucl. Phys.* **B793**, 34 (2008), [arXiv:0708.1023](#).
- [59] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of Type II Superstrings and the Moduli of Superconformal Field Theories*, *Int. J. Mod. Phys.* **A4**, 2475 (1989).
- [60] S. Ferrara and A. Marrani, *On the Moduli Space of non-BPS Attractors for $\mathcal{N}=2$ Symmetric Manifolds*, *Phys. Lett.* **B652**, 111 (2007), [arXiv:0706.1667](#).
- [61] S. Ferrara, R. Kallosh, A. Strominger, *$\mathcal{N}=2$ extremal black holes*, *Phys. Rev.* **D52**, 5412 (1995).
- [62] S. Ferrara, R. Kallosh, *Supersymmetry and attractors*, *Phys. Rev.* **D54**, 1514 (1996); S. Ferrara, R. Kallosh, *Universality of supersymmetric attractors*, *Phys. Rev.* **D54**, 1525 (1996).
- [63] A. Strominger, *Macroscopic entropy of $\mathcal{N}=2$ extremal black holes*, *Phys. Lett.* **B383**, 39 (1996).
- [64] S. Ferrara, G. W. Gibbons and R. Kallosh, *Black Holes and Critical Points in Moduli Space*, *Nucl. Phys.* **B500**, 75 (1997), [hep-th/9702103](#).
- [65] For reviews on black holes in superstring theory see *e.g.*: J. M. Maldacena, *Black-Holes in String Theory*, [hep-th/9607235](#); A. W. Peet, *TASI lectures on black holes in string theory*, [arXiv:hep-th/0008241](#); A. Dabholkar, *Black hole entropy and attractors*, *Class. Quant. Grav.* **23**, S957 (2006).
- [66] For recent reviews see: J. H. Schwarz, *Lectures on superstring and M-theory dualities*, *Nucl. Phys. Proc. Suppl.* **B55**, 1 (1997); M. J. Duff, *M-theory (the theory formerly known as strings)*, *Int. J. Mod. Phys.* **A11**, 5623 (1996); A. Sen, *Unification of string dualities*, *Nucl. Phys. Proc. Suppl.* **58**, 5 (1997).
- [67] J. H. Schwarz and A. Sen, *Duality symmetries of 4D heterotic strings*, *Phys. Lett.* **B312**, 105 (1993); J. H. Schwarz and A. Sen, *Duality Symmetrical Actions*, *Nucl. Phys.* **B411**, 35 (1994).
- [68] M. Gasperini, J. Maharana and G. Veneziano, *From trivial to non-trivial conformal string backgrounds via $O(d, d)$ transformations*, *Phys. Lett.* **B272**, 277 (1991); J. Maharana and J. H. Schwarz, *Noncompact Symmetries in String Theory*, *Nucl. Phys.* **B390**, 3 (1993).
- [69] E. Witten, *String Theory Dynamics in Various Dimensions*, *Nucl. Phys.* **B443**, 85 (1995).
- [70] J. H. Schwarz: *M-theory extensions of T duality*, [arXiv:hep-th/9601077](#); C. Vafa, *Evidence for F-theory*, *Nucl. Phys.* **B469**, 403 (1996).
- [71] K. Becker, M. Becker and J. H. Schwarz, *“String theory and M-theory: A modern introduction”*, Cambridge University Press (Cambridge, UK), 2007.
- [72] S. Ferrara and J. M. Maldacena, *Branes, central charges and U-duality invariant BPS conditions*, *Class. Quant. Grav.* **15**, 749 (1998), [hep-th/9706097](#).
- [73] H. Lu, C. N. Pope and K. S. Stelle, *Multiplet structures of BPS solitons*, *Class. Quant. Grav.* **15**, 537 (1998), [hep-th/9708109](#).
- [74] S. Bellucci, S. Ferrara, M. Günaydin and A. Marrani, *Charge orbits of symmetric special geometries and attractors*, *Int. J. Mod. Phys.* **A21**, 5043 (2006), [hep-th/0606209](#).

- [75] S. Ferrara and M. Günaydin, *Orbits and attractors for $\mathcal{N}=2$ Maxwell-Einstein supergravity theories in five dimensions*, Nucl. Phys. **B759**, 1 (2006), [hep-th/0606108](#).
- [76] L. Andrianopoli, R. D'Auria, S. Ferrara and M. Trigiante, *Extremal black holes in supergravity*, Lect. Notes Phys. **737**, 661 (2008), [hep-th/0611345](#).
- [77] S. Ferrara and A. Marrani, *$\mathcal{N}=8$ non-BPS Attractors, Fixed Scalars and Magic Supergravities*, Nucl. Phys. **B788**, 63 (2008), [arXiv:0705.3866](#).
- [78] S. Bellucci, S. Ferrara, R. Kallosh and A. Marrani, *Extremal Black Hole and Flux Vacua Attractors*, contribution to the Proceedings of *Winter School on Attractor Mechanism (SAM 2006)*, Frascati, Italy, 20-24 Mar 2006, [arXiv:0711.4547](#).
- [79] L. Andrianopoli, S. Ferrara, A. Marrani and M. Trigiante, *Non-BPS Attractors in 5d and 6d Extended Supergravity*, Nucl. Phys. **B795**, 428 (2008), [arXiv:0709.3488](#).
- [80] R. Kallosh, N. Sivanandam and M. Soroush, *Exact Attractive non-BPS STU Black Holes*, Phys. Rev. **D74**, 065008 (2006), [hep-th/0606263](#).
- [81] K. Hotta and T. Kubota, *Exact Solutions and the Attractor Mechanism in Non-BPS Black Holes*, Prog. Theor. Phys. **118N5**, 969 (2007), [arXiv:0707.4554](#).
- [82] E. G. Gimon, F. Larsen and J. Simon, *Black Holes in Supergravity: the non-BPS Branch*, JHEP **0801**, 040 (2008), [arXiv:0710.4967](#).
- [83] R.-G. Cai and D.-W. Pang, *A Note on exact solutions and attractor mechanism for non-BPS black holes*, JHEP **0801**, 046 (2008), [arXiv:0712.0217](#).
- [84] S. Bellucci, S. Ferrara, A. Marrani and A. Yeranyan, *stu Black Holes Unveiled*, [arXiv:0807.3503](#).
- [85] S. Bellucci, S. Ferrara, A. Marrani and A. Shcherbakov, *Quantum Lift of Non-BPS Flat Directions*, to appear.
- [86] M. Gogberashvili, *Rotations in the Space of Split Octonions*, [arXiv:0808.2496](#).
- [87] L. Andrianopoli, R. D'Auria and S. Ferrara, *U invariants, black hole entropy and fixed scalars*, Phys. Lett. **B403** 12 (1997), [hep-th/9703156](#).
- [88] L. Andrianopoli, R. D'Auria and S. Ferrara, *Five-dimensional U duality, black hole entropy and topological invariants*, Phys. Lett. **B411**, 39 (1997), [hep-th/9705024](#).
- [89] L. Andrianopoli, R. D'Auria and S. Ferrara, *Supersymmetry reduction of \mathcal{N} extended supergravities in four-dimensions*, JHEP **0203**, 025 (2002), [hep-th/0110277](#).
- [90] M. Bianchi and S. Ferrara, *Enriques and Octonionic Magic Supergravity Models*, JHEP **0802**, 054 (2008), [arXiv:0712.2976](#).
- [91] S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, *Second quantized mirror symmetry*, Phys. Lett. **B361**, 59 (1995), [hep-th/9505162](#).
- [92] J. A. Harvey and G. W. Moore, *Exact gravitational threshold correction in the FHSV model*, Phys. Rev. **D57**, 2329 (1998), [hep-th/9611176](#).
- [93] P. S. Aspinwall, *An $\mathcal{N}=2$ Dual Pair and a Phase Transition*, Nucl. Phys. **B460**, 57 (1996), [hep-th/9510142](#).
- [94] A. Klemm and M. Marino, *Counting BPS states on the Enriques Calabi-Yau*, Commun. Math. Phys. **280**, 27 (2008), [hep-th/0512227](#).
- [95] J. R. David, *On the dyon partition function in $\mathcal{N}=2$ theories*, JHEP **0802**, 025 (2008), [arXiv:0711.1971](#).
- [96] G. L. Cardoso, B. de Wit and S. Mahapatra, *Subleading and non-holomorphic corrections to $\mathcal{N}=2$ BPS black hole entropy*, [arXiv:0808.2627](#).

- [97] B. Kostant, *Minimal Coadjoint Orbits and Symplectic Induction* in : “*The Breadth of Symplectic and Poisson Geometry*”, Prog. Math. **232**, 391 (Birkhauser, Boston, 2003).

	IRGS <i>Classical Sequence</i> ($n, p, q \in \mathbb{N}$)	$rank$	$dim_{\mathbb{R}}$
\mathbf{I}_n (<i>A I</i>)	$\frac{SL(n, \mathbb{R})}{SO(n)}$	$n - 1$	$\frac{1}{2} (n - 1) (n + 2)$
\mathbf{II}_n (<i>A II</i>)	$\frac{SU^*(2n)}{USp(2n)}$	$n - 1$	$(n - 1) (2n + 1)$
$\mathbf{III}_{p,q}$ (<i>A III</i>)	$\frac{SU(p,q)}{SU(p) \otimes SU(q) \otimes U(1)}, \quad K$	$\min(p, q)$	$2pq$
$\mathbf{IV}_{p,q}$ (<i>BD I</i>)	$\frac{SO(p,q)}{SO(p) \otimes SO(q)}$	$\min(p, q)$	pq
\mathbf{V}_n (<i>D III</i>)	$\frac{SO^*(2n)}{U(n)}, \quad K$	$\left[\frac{n}{2} \right]$	$n (n - 1)$
\mathbf{VI}_n (<i>C I</i>)	$\frac{Sp(2n, \mathbb{R})}{U(n)}, \quad K$	n	$n (n + 1)$
$\mathbf{VII}_{p,q}$ (<i>C II</i>)	$\frac{USp(2p, 2q)}{USp(2p) \otimes USp(2q)}$	$\min(p, q)$	$4pq$
\mathbf{VIII}_G (<i>see text</i>)	$\frac{G_{\mathbb{C}}}{G_{\mathbb{R}}}$	$rank(G)$	$dim_{\mathbb{R}}(G)$

Table 1: **Classical** Infinite Sequences of Irreducible Riemannian Globally Symmetric Spaces of type I and type III (IRGS) (see *e.g.* Table II of [4] and Table 9.3 of [5]). The notation of Helgason's classification [4] is reported in brackets in the first column. Trivially, it holds that $\mathbf{III}_{p,q} = \mathbf{III}_{q,p}$, $\mathbf{IV}_{p,q} = \mathbf{IV}_{q,p}$ and $\mathbf{VII}_{p,q} = \mathbf{VII}_{q,p}$

	IRGS <i>Exceptional Case</i>	<i>rank</i>	<i>dim_ℝ</i>
1 (<i>E I</i>)	$\frac{E_{6(6)}}{USp(8)}$	6	42
2 (<i>E II</i>)	$\frac{E_{6(2)}}{SU(6) \otimes SU(2)}, H$	4	40
3 (<i>E III</i>)	$\frac{E_{6(-14)}}{SO(10) \otimes U(1)}, K$	2	32
4 (<i>E IV</i>)	$\frac{E_{6(-26)}}{F_4}$	2	26
5 (<i>E V</i>)	$\frac{E_{7(7)}}{SU(8)}$	7	70
6 (<i>E VI</i>)	$\frac{E_{7(-5)}}{SO(12) \otimes SU(2)}, H$	4	64
7 (<i>E VII</i>)	$\frac{E_{7(-25)}}{E_6 \otimes U(1)}, K$	3	54
8 (<i>E VIII</i>)	$\frac{E_{8(8)}}{SO(16)}$	8	128
9 (<i>E IX</i>)	$\frac{E_{8(-24)}}{E_7 \otimes SU(2)}, H$	4	112
10 (<i>F I</i>)	$\frac{F_{4(4)}}{USp(6) \otimes SU(2)}, H$	4	28
11 (<i>F II</i>)	$\frac{F_{4(-20)}}{SO(9)}$	1	16
12 (<i>G</i>)	$\frac{G_{2(2)}}{SU(2) \otimes SU(2)}, H$	2	8

Table 2: *Exceptional* Isolated Cases of IRGS (see *e.g.* Table II of [4] and Table 9.3 of [5]). The notation of Helgason’s classification [4] is reported in brackets in the first column. The subscript number in brackets denotes the *character* χ of the considered real form, defined as $\chi \equiv \# \text{ non-compact generators} - \# \text{ compact generators}$ (see *e.g.* Eq. (1.29), p. 332, as well as Table 9.3, of [5]). Concerning the compact form of (finite-dimensional) *exceptional* Lie groups, the following alternative notations exist: $G_2 \equiv G_{2(-14)}$, $F_4 \equiv F_{4(-52)}$, $E_6 \equiv E_{6(-78)}$, $E_7 \equiv E_{7(-133)}$ and $E_8 \equiv E_{8(-248)}$ (in other words, for a compact form $\chi = -\dim_{\mathbb{R}}$)

<i>Special Kähler</i> Symmetric Space	<i>Quaternionic</i> Symmetric Space
$\mathbf{III}_{1,n} \equiv \mathbb{CP}^n : \frac{SU(1,n)}{SU(n) \otimes U(1)}, \quad n \in \mathbb{N}$	$\mathbf{III}_{2,n+1} : \frac{SU(2,n+1)}{SU(n+1) \otimes SU(2) \otimes U(1)}, \quad n \in \mathbb{N} \cup \{0\}$
$\mathbf{III}_{1,1} \otimes \mathbf{IV}_{2,n} : \frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n)}{SO(n) \otimes U(1)},$ $n \in \mathbb{N} \quad (\mathbb{R} \oplus \mathbf{\Gamma}_{n-1,1})$	$\mathbf{IV}_{4,n+2} : \frac{SO(4,n+2)}{SO(n+2) \otimes SO(4)},$ $n \in \mathbb{N} \cup \{0, -1\} \quad (\mathbb{R} \oplus \mathbf{\Gamma}_{n-1,1})$
$\mathbf{III}_{1,1} : \frac{SU(1,1)}{U(1)} \quad (\mathbb{R})$	$\mathbf{12} : \frac{G_{2(2)}}{SO(4)} \quad (\mathbb{R})$
$\mathbf{VI}_3 : \frac{Sp(6, \mathbb{R})}{SU(3) \otimes U(1)} \quad (J_3^{\mathbb{R}})$	$\mathbf{10} : \frac{F_{4(4)}}{USp(6) \otimes SU(2)} \quad (J_3^{\mathbb{R}})$
$\mathbf{III}_{3,3} : \frac{SU(3,3)}{SU(3) \otimes SU(3) \otimes U(1)} \quad (J_3^{\mathbb{C}})$	$\mathbf{2} : \frac{E_{6(2)}}{SU(6) \otimes SU(2)} \quad (J_3^{\mathbb{C}})$
$\mathbf{V}_6 : \frac{SO^*(12)}{SU(6) \otimes U(1)} \quad (J_3^{\mathbb{H}}, \mathcal{N} = 2 \Leftrightarrow \mathcal{N} = 6)$	$\mathbf{6} : \frac{E_{7(-5)}}{SO(12) \otimes SU(2)} \quad (J_3^{\mathbb{H}}, \mathcal{N} = 4 \Leftrightarrow \mathcal{N} = 12)$
$\mathbf{7} : \frac{E_{7(-25)}}{E_6 \otimes SO(2)} \quad (J_3^{\mathbb{O}})$	$\mathbf{9} : \frac{E_{8(-24)}}{E_7 \otimes SU(2)} \quad (J_3^{\mathbb{O}})$

Table 3: $\mathcal{N}=2$, $d=4$ symmetric special Kähler vector multiplets' scalar manifolds and the corresponding symmetric quaternionic spaces, obtained through *c-map* [59]. In general, starting from a special Kähler geometry with $\dim_{\mathbb{C}} = n$, the *c-map* generates a quaternionic manifold with $\dim_{\mathbb{H}} = n + 1$ [59]. If any, the related *Jordan algebras of degree three* are reported in brackets throughout (the notation of [55] is used, see also Table 2 therein). By defining $A \equiv \dim_{\mathbb{R}} \mathbb{A}$ ($= 1, 2, 4, 8$ for $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ respectively), the *complex* dimension of the $\mathcal{N}=2$, $d=4$ symmetric special Kähler manifolds based on $J_3^{\mathbb{A}}$ is $3A + 3$ [60]. Thus, the *quaternionic* dimension of the corresponding $\mathcal{N}=2$, $d=4$ symmetric quaternionic manifolds obtained through *c-map* is $3A + 4$ [59, 90]

<i>Associated Jordan Algebra of degree three (in $d = 5$)</i>	$\frac{\hat{H}}{\hat{h}}$
$\mathbb{R} \oplus \mathbf{I}_{n-1,1}, n \in \mathbb{N}$	$SO(1,1) \otimes \mathbf{IV}_{1,n-1} : SO(1,1) \otimes \frac{SO(1,n-1)}{SO(n-1)}$
$J_3^{\mathbb{O}}$	$\mathbf{4} : \frac{E_{6(-26)}}{F_4}$
$J_3^{\mathbb{H}}$	$\mathbf{II}_3 : \frac{SU^*(6)}{USp(6)}$
$J_3^{\mathbb{C}}$	$\mathbf{VIII}_{SU(3)} : \frac{SL(3,\mathbb{C})}{SU(3)}$
$J_3^{\mathbb{R}}$	$\mathbf{I}_3 : \frac{SL(3,\mathbb{R})}{SO(3)}$

Table 4: *Moduli spaces* of non-BPS $Z \neq 0$ critical points of $V_{BH,\mathcal{N}=2}$ in $\mathcal{N}=2$, $d=4$ special Kähler symmetric vector multiplets' scalar manifolds [60]. They are nothing but the $\mathcal{N}=2$, $d=5$ real special symmetric vector multiplets' scalar manifolds. \hat{H} is the *non-compact* stabilizer of the corresponding supporting *charge orbit* $\mathcal{O}_{non-BPS,Z \neq 0}$ [74], and \hat{h} is its *maximal compact subgroup* (with symmetric embedding). As observed in [60], the *real* dimension of $\mathcal{N}=2$, $d=5$ real special symmetric manifolds based on $J_3^{\mathbb{A}}$ is $3A+2$

<i>Jordan Algebra of degree three (of the corresponding scalar manifold in $d = 4$)</i>	$\frac{\tilde{H}}{\tilde{h}} = \frac{\tilde{H}}{\tilde{h}' \otimes U(1)}$
—	$\mathbf{III}_{1,n-1} : \frac{SU(1,n-1)}{U(1) \otimes SU(n-1)}, \quad SK \quad (H \text{ for } n = 3)$
$\mathbb{R} \oplus \Gamma_{n-1,1}, \quad n \geq 3$	$\mathbf{IV}_{2,n-2} : \frac{SO(2,n-2)}{SO(2) \otimes SO(n-2)} \quad (H \text{ for } n = 6)$
$J_3^{\mathbb{O}}$	$\mathbf{3} : \frac{E_{6(-14)}}{SO(10) \otimes U(1)}$
$J_3^{\mathbb{H}}$	$\mathbf{III}_{4,2} : \frac{SU(4,2)}{SU(4) \otimes SU(2) \otimes U(1)}, \quad H$
$J_3^{\mathbb{C}}$	$(\mathbf{III}_{2,1})^2 : \frac{SU(2,1)}{SU(2) \otimes U(1)} \otimes \frac{SU(1,2)}{SU(2) \otimes U(1)}, \quad SK, H$
$J_3^{\mathbb{R}}$	$\mathbf{III}_{2,1} : \frac{SU(2,1)}{SU(2) \otimes U(1)}, \quad SK, H$

Table 5: *Moduli spaces* of non-BPS $Z = 0$ critical points of $V_{BH, \mathcal{N}=2}$ in $\mathcal{N} = 2$, $d = 4$ special Kähler symmetric vector multiplets' scalar manifolds [60]. Unless otherwise noted, they are non-special Kähler symmetric manifolds. \tilde{H} is the *non-compact* stabilizer of the corresponding supporting *charge orbit* $\mathcal{O}_{non-BPS, Z=0}$ [74], and \tilde{h} is its *maximal compact subgroup* (with symmetric embedding). As observed in [60], the *complex* dimension of the *moduli spaces* of non-BPS $Z = 0$ critical points of $V_{BH, \mathcal{N}=2}$ in $\mathcal{N} = 2$, $d = 4$ special Kähler symmetric manifolds based on $J_3^{\mathbb{A}}$ is $2A$

\mathcal{N}	$G_{\mathcal{N},4}/H_{\mathcal{N},4}$
3	$\mathbf{III}_{3,n} : \frac{SU(3,n)}{SU(3) \otimes SU(n) \otimes U(1)}, \quad n \in \mathbb{N}$
4	$\mathbf{III}_{1,1} \otimes IV_{6,n} : \frac{SU(1,1)}{U(1)} \otimes \frac{SO(6,n)}{SO(6) \otimes SO(n)}, \quad n \in \mathbb{N} \cup \{0\} \quad (\mathbb{R} \oplus \mathbf{F}_{n-1,5})$
5	$\mathbf{III}_{1,5} : \frac{SU(1,5)}{SU(5) \otimes U(1)} \quad (M_{1,2}(\mathbb{O}))$
6	$\mathbf{V}_6 : \frac{SO^*(12)}{SU(6) \otimes U(1)} \quad (J_3^{\mathbb{H}})$
8	$\mathbf{5} : \frac{E_{7(7)}}{SU(8)} \quad (J_3^{\mathbb{O}_s})$

Table 6: **Scalar manifolds of $\mathcal{N} \geq 3$, $d = 4$ supergravities.** Notice that the scalar manifold of $\mathcal{N} = 6$ supergravity coincides with the one of $\mathcal{N} = 2$ supergravity based on $J_3^{\mathbb{H}}$ (see Table 3)

\mathcal{N}	$\frac{1}{\mathcal{N}}$ -BPS moduli space $\frac{\mathcal{H}}{\mathfrak{h}}$	non-BPS, $Z_{AB} \neq 0$ moduli space $\frac{\widehat{\mathcal{H}}}{\mathfrak{h}}$	non-BPS, $Z_{AB} = 0$ moduli space $\frac{\widetilde{\mathcal{H}}}{\mathfrak{h}}$
3	$\text{III}_{2,n} : \frac{SU(2,n)}{SU(2) \otimes SU(n) \otimes U(1)},$ $n \in \mathbb{N}$	—	$\text{III}_{3,n-1} : \frac{SU(3,n-1)}{SU(3) \otimes SU(n-1) \otimes U(1)},$ $n \geq 2$
4	$\text{IV}_{4,n} : \frac{SO(4,n)}{SO(4) \otimes SO(n)},$ $n \in \mathbb{N}$	$SO(1,1) \otimes \text{IV}_{5,n-1} :$ $SO(1,1) \otimes \frac{SO(5,n-1)}{SO(5) \otimes SO(n-1)},$ $n \in \mathbb{N}$	$\text{IV}_{6,n-2} : \frac{SO(6,n-2)}{SO(6) \otimes SO(n-2)},$ $n \geq 3$
5	$\text{III}_{2,1} : \frac{SU(2,1)}{SU(2) \otimes U(1)}$	—	—
6	$\text{III}_{4,2} : \frac{SU(4,2)}{SU(4) \otimes SU(2) \otimes U(1)}$	$\text{II}_3 : \frac{SU^*(6)}{USp(6)}$	—
8	$\mathbf{2} : \frac{E_{6(2)}}{SU(6) \otimes SU(2)}$	$\mathbf{1} : \frac{E_{6(6)}}{USp(8)}$	—

Table 7: *Moduli spaces* of extremal black hole attractors with *non-vanishing* classical entropy in $3 \leq \mathcal{N} \leq 8$, $d = 4$ *supergravities* [87, 76, 77, 60, 78]. (see Table 1 of [78]). \mathfrak{h} , $\widehat{\mathfrak{h}}$ and $\widetilde{\mathfrak{h}}$ respectively are the *maximal compact subgroups* (with symmetric embedding) of \mathcal{H} , $\widehat{\mathcal{H}}$ and $\widetilde{\mathcal{H}}$, which in turn are the *non-compact stabilizers* of the corresponding supporting *charge orbits* $\mathcal{O}_{1/\mathcal{N}-\text{BPS}}$, $\mathcal{O}_{\text{non-BPS}, Z_{AB} \neq 0}$ and $\mathcal{O}_{\text{non-BPS}, Z_{AB} = 0}$, respectively [44, 74, 56, 76, 77, 60, 78](see Table 1 of [78])

<i>Jordan Algebra of degree three (of the corresponding scalar manifold in $d = 5$)</i>	$\frac{\tilde{H}_5}{K_5}$
$\mathbb{R} \oplus \Gamma_{n-1,1}, \quad n \geq 3$	$\mathbf{IV}_{1,n-2} : \frac{SO(1,n-2)}{SO(n-2)}$
$J_3^{\mathbb{O}}$	$\mathbf{11} : \frac{F_{4(-20)}}{SO(9)}$
$J_3^{\mathbb{H}}$	$\mathbf{VII}_{1,2} : \frac{USp(4,2)}{USp(4) \otimes USp(2)}$
$J_3^{\mathbb{C}}$	$\mathbf{III}_{2,1} : \frac{SU(2,1)}{SU(2) \otimes U(1)}$
$J_3^{\mathbb{R}}$	$\mathbf{I}_2 : \frac{SL(2,\mathbb{R})}{SO(2)}$

Table 8: *Moduli spaces* of non-BPS ($Z \neq 0$) critical points of $V_{BH,\mathcal{N}=2}$ in $\mathcal{N} = 2$, $d = 5$ real special symmetric vector multiplets' scalar manifolds [60]. \tilde{H}_5 is the *non-compact* stabilizer of the corresponding supporting *charge orbit* $\mathcal{O}_{non-BPS}$ [60], and \tilde{K}_5 is its *maximal compact subgroup* (with symmetric embedding). As observed in [60], the *real* dimension of the *moduli spaces* of non-BPS ($Z \neq 0$) critical points of $V_{BH,\mathcal{N}=2}$ in $\mathcal{N} = 2$, $d = 5$ real special symmetric manifolds based on $J_3^{\mathbb{A}}$ is $2A$, and the stabilizer of such *moduli spaces* contains the group $Spin(1+A)$

\mathcal{N}	$G_{\mathcal{N},5}/H_{\mathcal{N},5}$
4	$SO(1,1) \otimes \mathbf{IV}_{5,n-1} : SO(1,1) \otimes \frac{SO(5,n-1)}{SO(5) \otimes SO(n-1)}, \quad n \in \mathbb{N} \quad (\mathbb{R} \oplus \Gamma_{n-1,5})$
6	$\mathbf{II}_3 : \frac{SU^*(6)}{USp(6)} \quad (J_3^{\mathbb{H}})$
8	$\mathbf{1} : \frac{E_{6(6)}}{USp(8)} \quad (J_3^{\mathbb{O}_s})$

Table 9: *Scalar manifolds* of $\mathcal{N} > 2$, $d = 5$ *supergravities*. Notice that, also for $d = 5$, the scalar manifold of $\mathcal{N} = 6$ *supergravity* coincides with the one of $\mathcal{N} = 2$ *supergravity* based on $J_3^{\mathbb{H}}$ (see Table 4)

\mathcal{N}	$\frac{1}{\mathcal{N}}$ -BPS moduli space $\frac{\mathcal{H}_5}{\mathfrak{h}_5}$	non-BPS ($Z_{AB} \neq 0$) moduli space $\frac{\hat{\mathcal{H}}_5}{\hat{\mathfrak{h}}_5}$
4	$\mathbf{IV}_{4,n-1} : \frac{SO(4,n-1)}{SO(4) \otimes SO(n-1)}, \quad n \geq 2$	$\mathbf{IV}_{5,n-2} : \frac{SO(5,n-2)}{SO(5) \otimes SO(n-2)}, \quad n \geq 3$
6	$\mathbf{VII}_{1,2} : \frac{USp(4,2)}{USp(4) \otimes USp(2)}$	—
8	$\mathbf{10} : \frac{F_{4(4)}}{USp(6) \otimes USp(2)}$	—

Table 10: *Moduli spaces* of extremal black hole attractors with *non-vanishing* classical entropy in $4 \leq \mathcal{N} \leq 8$, $d = 5$ *supergravities* [77, 60, 79]. \mathfrak{h}_5 and $\hat{\mathfrak{h}}_5$ respectively are the *maximal compact subgroups* (with symmetric embedding) of \mathcal{H}_5 and $\hat{\mathcal{H}}_5$, which in turn are the *non-compact stabilizers* of the corresponding supporting *charge orbits* $\mathcal{O}_{1/\mathcal{N}-BPS}$ and $\mathcal{O}_{non-BPS}$, respectively [44, 75, 56, 77, 60, 79]

\mathcal{N}	$G_{\mathcal{N},3}/H_{\mathcal{N},3}$
5	$\mathbf{VII}_{2,n} : \frac{USp(4,2n)}{USp(4) \otimes USp(2n)}, \quad n \in \mathbb{N}$
6	$\mathbf{III}_{4,n} : \frac{SU(4,n)}{SU(4) \otimes SU(n) \otimes U(1)}, \quad n \in \mathbb{N}$
8	$\mathbf{IV}_{8,n+2} : \frac{SO(8,n+2)}{SO(8) \otimes SO(n+2)}, \quad n \in \mathbb{N} \cup \{0, -1\} \quad (\mathbb{R} \oplus \mathbf{\Gamma}_{n-1,5})$
9	$\mathbf{11} : \frac{F_{4(-20)}}{SO(9)}$
10	$\mathbf{3} : \frac{E_{6(-14)}}{SO(10) \otimes SO(2)} \quad (M_{1,2}(\mathbb{O}))$
12	$\mathbf{6} : \frac{E_{7(-5)}}{SO(12) \otimes SU(2)} \quad (J_3^{\mathbb{H}})$
16	$\mathbf{8} : \frac{E_{8(8)}}{SO(16)} \quad (J_3^{\mathbb{O}_s})$

Table 11: *Scalar manifolds* of $\mathcal{N} \geq 5$, $d = 3$ *supergravities* [29]. Notice that the scalar manifold of $\mathcal{N} = 12$ supergravity coincides with the one of ($\mathcal{N} = 4$) *supergravity* based on $J_3^{\mathbb{H}}$ (see Table 3)

\mathbb{A}	$\mathcal{M}_{5,J_3^{\mathbb{A}}}$	$\mathcal{B}_{5,A}$	$\mathcal{F}_{5,J_3^{\mathbb{A}}}$
\mathbb{O}	$\frac{E_6(-26)}{F_4}$	$SO(1,1) \otimes \frac{SO(1,9)}{SO(9)}$	$\frac{F_4(-20)}{SO(9)}$
\mathbb{H}	$\frac{SU^*(6)}{USp(6)}$	$SO(1,1) \otimes \frac{SO(1,5)}{SO(5)}$	$\frac{USp(4,2)}{USp(4) \otimes USp(2)}$
\mathbb{C}	$\frac{SL(3,\mathbb{C})}{SU(3)}$	$SO(1,1) \otimes \frac{SO(1,3)}{SO(3)}$	$\frac{SU(2,1)}{SU(2) \otimes U(1)}$
\mathbb{R}	$\frac{SL(3,\mathbb{R})}{SO(3)}$	$SO(1,1) \otimes \frac{SO(1,2)}{SO(2)}$	$\frac{SL(2,\mathbb{R})}{SO(2)}$

Table 12: $d = 5$ *Exceptional sequence* [90]. Trivially, all manifolds of such a Table are real, and they also all are RS but the sequence $\left\{ \mathcal{F}_{5,J_3^{\mathbb{A}}} \right\}_{\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}}$, which is new

\mathbb{A}	$\mathcal{M}_{4,J_3^{\mathbb{A}}}$	$\mathcal{B}_{4,A}$	$\mathcal{F}_{4,J_3^{\mathbb{A}}}$
\mathbb{O}	$\frac{E_7(-25)}{E_6 \otimes SO(2)}$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,10)}{SO(10) \otimes U(1)}$	$\frac{E_6(-14)}{SO(10) \otimes U(1)}$
\mathbb{H}	$\frac{SO^*(12)}{SU(6) \otimes U(1)}$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,6)}{SO(6) \otimes U(1)}$	$\frac{SU(4,2)}{SU(4) \otimes SU(2) \otimes U(1)}$
\mathbb{C}	$\frac{SU(3,3)}{SU(3) \otimes SU(3) \otimes U(1)}$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,4)}{SO(4) \otimes U(1)}$	$\frac{SU(2,1)}{SU(2) \otimes U(1)} \otimes \frac{SU(1,2)}{SU(2) \otimes U(1)}$
\mathbb{R}	$\frac{Sp(6,\mathbb{R})}{SU(3) \otimes U(1)}$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,3)}{SO(3) \otimes U(1)}$	$\frac{SU(2,1)}{SU(2) \otimes U(1)}$

Table 13: $d = 4$ *Exceptional sequence* [90]. All manifolds of such a Table are K, and they also all are SK but $\mathcal{F}_{4,J_3^{\mathbb{O}}}$ and $\mathcal{F}_{4,J_3^{\mathbb{H}}}$. The sequence $\left\{ \mathcal{F}_{4,J_3^{\mathbb{A}}} \right\}_{\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}}$ has been obtained in [58] through *constrained instantons*

\mathbb{A}	$\mathcal{M}_{3,J_3^{\mathbb{A}}}$	$\mathcal{B}_{3,A}$	$\mathcal{F}_{3,J_3^{\mathbb{A}}}$
\mathbb{O}	$\frac{E_{8(-24)}}{E_7 \otimes SU(2)}$	$\frac{SO(4,12)}{SO(12) \otimes SO(4)}$	$\mathbf{6} : \frac{E_{7(-5)}}{SO(12) \otimes SU(2)}, H$
\mathbb{H}	$\frac{E_{7(-5)}}{SO(12) \otimes SU(2)}$	$\frac{SO(4,8)}{SO(8) \otimes SO(4)}$	$\mathbf{IV}_{4,8} : \frac{SO(4,8)}{SO(8) \otimes SO(4)}, H$
\mathbb{C}	$\frac{E_{6(2)}}{SU(6) \otimes SU(2)}$	$\frac{SO(4,6)}{SO(6) \otimes SO(4)}$	$\mathbf{III}_{4,2} : \frac{SU(4,2)}{SU(4) \otimes SU(2) \otimes U(1)}, H$
\mathbb{R}	$\frac{F_{4(4)}}{USp(6) \otimes SU(2)}$	$\frac{SO(4,5)}{SO(5) \otimes SO(4)}$	$\mathbf{VII}_{1,2} \equiv \mathbb{HP}^2 : \frac{USp(4,2)}{USp(4) \otimes USp(2)}, H$

Table 14: $d = 3$ *Exceptional sequence* [90]. All manifolds of such a Table are H. The sequence $\left\{ \mathcal{F}_{3,J_3^{\mathbb{A}}} \right\}_{\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}}$ is new

$\frac{G}{K}$	X	\mathcal{X}	$\dim_{\mathbb{R}}(X)$	$\text{rank}(X)$	$J(X)$
$\frac{G_{2(2)}}{SU(2) \otimes SU(2)}$	$\left(\frac{SU(2)}{U(1)} \right)^2$	$\left(\frac{SU(1,1)}{U(1)} \right)^2$	4	2	$\mathbb{R} \oplus \mathbb{R}$
$\frac{F_{4(4)}}{SU(2) \otimes USp(6)}$	$\frac{SU(2)}{U(1)} \otimes \frac{USp(6)}{U(3)}$	$\frac{SU(1,1)}{U(1)} \otimes \frac{Sp(6,\mathbb{R})}{U(3)}$	14	3	$\mathbb{R} \oplus J_3^{\mathbb{R}}$
$\frac{E_{6(6)}}{USp(8)}$	$\frac{USp(8)}{U(4)}$	$\frac{Sp(8,\mathbb{R})}{U(4)}$	20	4	$J_4^{\mathbb{R}}$
$\frac{E_{7(7)}}{SU(8)}$	$\frac{SU(8)}{SU(4) \otimes SU(4) \otimes U(1)}$	$\frac{SU(4,4)}{SU(4) \otimes SU(4) \otimes U(1)}$	32	4	$J_4^{\mathbb{C}}$
$\frac{E_{8(8)}}{SO(16)}$	$\frac{SO(16)}{U(8)}$	$\frac{SO^*(16)}{U(8)}$	56	4	$J_4^{\mathbb{H}}$

Table 15: Some particular IRGS $\frac{G}{K}$ and their associated *compact* spaces X (along with their *unique non-compact* (I)RGS \mathcal{X}), and the corresponding Jordan algebra $J(X)$. The relation among $\frac{G}{K}$ and X is based on *minimal coadjoint orbits* and *symplectic induction*, and it is due to Kostant [97]